STATISTICAL MECHANICS OF THE SELF-GRAVITATING GAS WITH TWO OR MORE KINDS OF PARTICLES

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Abstract

We study the statistical mechanics of the self-gravitating gas at thermal equilibrium with two kinds of particles. We start from the partition function in the canonical ensemble which we express as a functional integral over the densities of the two kinds of particles for a large number of particles. The system is shown to possess an infinite volume limit when $(N_1, N_2, V) \to \infty$, keeping $N_1/V^{\frac{1}{3}}$ and $N_2/V^{\frac{1}{3}}$ fixed. The saddle point approximation becomes here exact for $(N_1, N_2, V) \to \infty$. It provides a nonlinear differential equation on the densities of each kind of particles. For the spherically symmetric case, we compute the densities as functions of two dimensionless physical parameters: $\eta_1 = \frac{Gm_1^2 N_1}{V^{\frac{1}{3}} T}$ and $\eta_2 = \frac{Gm_2^2 N_2}{V^{\frac{1}{3}} T}$ (where G is Newton's constant, m_1 and m_2 the masses of the two kinds of particles and T the temperature). According to the values of η_1 and η_2 the system can be either in a gaseous phase or in a highly condensed phase. The gaseous phase is stable for η_1 and η_2 between the origin and their collapse values. We have thus generalized the well-known isothermal sphere for two kinds of particles. The gas is inhomogeneous and the mass $M(R)$ inside a sphere of radius R scales with R as $M(R) \propto R^d$ suggesting a fractal structure. The value of d depends in general on η_1 and η_2 except on the critical line for the canonical ensemble in the (η_1, η_2) plane where it takes the universal value $d \simeq 1.6$ for all values of N_1/N_2. The equation of state is computed. It is found to be **locally** a perfect gas equation of state. The thermodynamic functions (free energy, energy, entropy) are expressed and plotted as functions of η_1 and η_2. They exhibit a square root Riemann sheet with the branch points on the critical canonical line. The behaviour of the energy and the specific heat at the critical line is computed. This treatment is further generalized to the self-gravitating gas with n-types of particles.

Contents

1 Introduction

The self-gravitating gases have remarkable physical properties due to the long range nature of the gravitational force. They are not homogeneous even at thermal equilibrium. This fundamental inhomogeneity character suggested that fractal structures can arise in a self-interacting gravitational gas[1, 2, 4].

Self-gravitating gases are used to describe cold clouds in the interstellar medium as well as the large scale structure of galaxies. In both cases, self-gravitating gases provide scaling laws for the mass distribution with Haussdorf dimensions compatible with the observations[1, 2, 4].

All particles have the same mass in the self-gravitating gases in thermal equilibrium considered till now both in the hydrostatic approach [3] and in the statistical mechanics approach[4]. We study in the present paper the statistical mechanics of a non-relativistic gas with two kinds of particles with masses m_1 and m_2. Such a system, besides its own physical interest, has obvious astrophysical motivations since cold clouds in the galaxy are formed typically by several kinds of particles. For example, hydrogen and helium.

Let us begin by briefly recalling some results about the statistical mechanics of the self-gravitating gas with one kind of particles [4]. It is a gas of N non-relativistic particles of mass m interacting through Newtonian gravity. The gas is in a volume V and in a thermal bath at temperature T. In the usual thermodynamic limit ($N, V \to \infty$ and N/V is fixed) the gaseous phase is not stable and the system collapses in a very dense phase. In the dilute limit $N, V \to \infty$ and $N/V^{\frac{1}{3}}$ fixed the system can exist in the gaseous phase. This is a dilute limit since the average density $\frac{N}{V}$ goes as $V^{-\frac{2}{3}} \to 0$. The relevant

physical parameter of the system is $\eta = \frac{Gm^2N}{LT}$ with Newton's constant G and the length $L \equiv V^{\frac{1}{3}}$. η is the ratio of the characteristic gravitational energy $\frac{Gm^2N}{L}$ and the kinetic energy T of the gas. For $\eta = 0$ the ideal gas is recovered. From $\eta = 0$ till a critical value $\eta_0 = 2.43450\ldots$ the gaseous phase is stable in the canonical ensemble. When η reaches the value η_0 the gas collapses in a very dense phase. The velocity of sound becomes imaginary at this point triggering unstabilities that lead to the collapse of the gas[4]. The saddle point approximation applies between $\eta = 0$ and the point $\eta^C = 2.517551\ldots$ (associated to the Jeans instability) where the determinant of small fluctuations is positive. At η^C the determinant of small fluctuations vanishes and the saddle point approximation breaks down[4]. Beyond η^C the gaseous phase is stable and the mean field approximation holds in the microcanonical ensemble. Solving the saddle point equation in the spherically symmetric case allows to obtain the particle density and the thermodynamic functions as functions of the physical parameter η. As shown in ref.[4] the mean field approach is equivalent to the hydrostatic description[3] provided the ideal gas equation of state is postulated in the latter approach. The mass distribution turns to exhibit a scaling behaviour as a function of R[4].

We consider in this paper the self-gravitating gas with two kinds of particles in the canonical ensemble. We recast the partition function as a functional integral over the densities of particles of each kind when the number of particles is large. The statistical weight for each configuration of densities turns to be the exponential of an 'effective action' which is proportional to the number of particles. Therefore, we can use the saddle point approximation in the thermodynamic limit to evaluate the partition function. The 'effective action' turns to be the free energy as a functional of the particle densities.

When the saddle point provides a minimum of the free energy, the density solution of the saddle point equation is the most probable. It is certainly exact for an infinite number of particles, since the minimized free energy exponentially dominates the partition function. That is, the mean field theory defined by the saddle point becomes exact for an infinite number of particles. The mean field approximation for the canonical ensemble ceases to be valid on a critical line in the (η_1, η_2) plane (see below). Beyond this critical line the saddle point is not a minimum of the free energy and it fails to reproduce the physics of the system.

More precisely, we consider N_1 particles of mass m_1 and N_2 particles of mass m_2 interacting through Newtonian gravity in a volume V and in a thermal equilibrium at temperature T. By analogy with the self-gravitating gas with one kind of particles, we consider the dilute thermodynamic limit:

$$N_1, \ N_2, \ V \to \infty \quad \text{keeping} \quad \frac{N_1}{V^{\frac{1}{3}}} \quad \text{and} \quad \frac{N_2}{V^{\frac{1}{3}}} \quad \text{fixed} \tag{1}$$

The two relevant physical parameter are here

$$\eta_1 = \frac{Gm_1^2 N_1}{L\,T} \quad , \quad \eta_2 = \frac{Gm_2^2 N_2}{L\,T} \tag{2}$$

where $L \equiv V^{\frac{1}{3}}$ for a cubic geometry. Notice that eq.(1) implies that the ratio N_1/N_2 stays fixed for $N_1, \ N_2, \ V \to \infty$.

The self-gravitating gas with two kinds of particle behaves as a perfect gas in the extremely diluted limit $\eta_1 \to 0$ and $\eta_2 \to 0$. When η_1 and/or η_2 grow, the gas becomes denser till it collapses into a very dense phase when η_1 and η_2 reach the collapse line for

the canonical ensemble in the (η_1, η_2) plane. By analogy with the gas with one kind of particles [4] we expect the collapse line to be very close and below the critical line in the (η_1, η_2) plane. The gaseous phase keeps stable in the microcanonical ensemble beyond this critical line.

We find here that the saddle point equations are two coupled non-linear differential equations for the densities of the two kinds of particles: $\rho_1(\mathbf{x}), \rho_2(\mathbf{x})$. We succeed to reduce these equations to a single non-linear differential equation. We solve it in the spherically symmetric case. We express the densities as functions of the physical parameters η_1 and η_2. We thus find the isothermal sphere with two types of particles.

We compute the mass inside a sphere of radius R centered at the origin and show that it scales with a Haussdorf dimension d. d decreases with η_1^R and η_2^R from the value $d = 3$ for the ideal homogeneous gas till $d \approx 1.6$ in the canonical critical line. The Haussdorf dimension keeps decreasing beyond the canonical critical line in the stable phase of the microcanonical ensemble. d at the canonical critical line turns to be **independent** of the ratio η_1^R/η_2^R and coinciding within the numerical precision with $d \approx 1.6$ for the canonical critical point of the gas with one kind of particles [4]. This indicates that the Haussdorf dimension at the canonical critical line is an universal value, independent of the gas composition.

The dependence of the critical values of the parameters η_1 and η_2 with the number of particles ratio is computed. The thermodynamic functions (free energy, energy, entropy, local pressure and pressure contrast) are expressed as functions of the physical parameter η_1 and η_2. The pressure contrast [ratio between the pressure at the origin and the pressure at the boundary] turns to be **lower** for this mixture of particles than for the gas with one kind of particles.

We compute the pressure at a point $\mathbf{r}$ of the gas and show that it **locally** obeys the equation of state of an ideal gas,

$$P(\mathbf{r}) = \frac{N_1 T}{V} \, \rho_1(\mathbf{r}) + \frac{N_2 T}{V} \, \rho_2(\mathbf{r}) \ . \tag{3}$$

Since the gas is inhomogeneous, the pressure acting on any non-infinitesimal volume of the gas does not obey the ideal equation of state. (This was already the case for a gas with all particles of equal mass[4]). We plot in figs. 3-5 the pressure of the gas at the surface.

The mean field equations have a straightforward hydrostatic interpretation. We show that the mean field equations derived from the partition function are equivalent to the hydrostatic equilibrium equations **provided** that the ideal equation of state is postulated in the latter approach. We stress that we give here a **microscopic derivation** of the equation of state (3) from the partition function.

We then consider a self-gravitating gas formed by n kinds of particles with different masses. The mean field equations are derived and shown to reduce to a single non-linear differential equation.

This paper is organized as follows, In section II we present the statistical mechanics of the self-gravitating gas with two kinds of particles in the canonical ensemble, in sec. III we present the main thermodynamic magnitudes, the equation of state and the scaling behaviour of the particle distribution for spherical symmetry. In sec. IV we present the generalization for n kinds of particles. The Appendixes contain relevant mathematical developments and the hydrostatic approach to a self-gravitating gas with two kinds of particles.

2 Statistical mechanics and mean field theory for the self-gravitating gas

We present here the partition function for the self-gravitating gas with N_1 particles of mass m_1 and N_2 particles of mass m_2 inside a finite volume V and derive the mean field approach to it.

2.1 The canonical ensemble

We study the statistical mechanics of the self-gravitating gas with two kinds of particles in the canonical ensemble. The Hamiltonian of the system is

$$
\begin{aligned}
H = \sum_{i=1}^{N_1} \frac{\mathbf{p_{1,i}^2}}{2m_1} &+ \sum_{i=1}^{N_2} \frac{\mathbf{p_{2,i}^2}}{2m_2} \\
- \sum_{1 \leq i < j \leq N_1} \frac{G\, m_1^2}{|\mathbf{q_{1,i}} - \mathbf{q_{1,j}}|} &- \sum_{1 \leq i < j \leq N_2} \frac{G\, m_2^2}{|\mathbf{q_{2,i}} - \mathbf{q_{2,j}}|} - \sum_{1 \leq i \leq N_1, 1 \leq j \leq N_2} \frac{G\, m_1 m_2}{|\mathbf{q_{1,i}} - \mathbf{q_{2,j}}|} \quad .
\end{aligned}
$$

Here, $\mathbf{p_{1,i}}$ and $\mathbf{q_{1,i}}$ are the momenta and the coordinates of the particles of mass m_1. $\mathbf{p_{2,i}}$ and $\mathbf{q_{2,i}}$ are the momenta and the coordinates of the particles of mass m_2. Therefore, the classical partition function of the gas is

$$
Z(T, N_1, N_2, V) = \frac{1}{N_1! N_2!} \int \ldots \int \prod_{l=1}^{l=N_1} \frac{d^3\mathbf{p_{1,l}}\, d^3\mathbf{q_{1,l}}}{(2\pi)^3} \prod_{l=1}^{l=N_2} \frac{d^3\mathbf{p_{2,l}}\, d^3\mathbf{q_{2,l}}}{(2\pi)^3}\, e^{-\frac{H}{T}} \quad .
$$

It is convenient to introduce the dimensionless coordinates variables $\mathbf{r_{1,l}} = L\, \mathbf{q_{1,l}}$ and $\mathbf{r_{2,l}} = L\, \mathbf{q_{2,l}}$. The momenta integrals are computed straightforwardly. Hence the partition function becomes the product of the partition function of perfect gases with masses m_1 and m_2 times the coordinate integral Z_{int}.

$$
Z = \frac{V^{N_1}}{N_1!} \left(\frac{m_1 T}{2\pi}\right)^{3N_1/2} \frac{V^{N_2}}{N_2!} \left(\frac{m_2 T}{2\pi}\right)^{3N_2/2} Z_{int} ,
$$

where

$$
Z_{int} = \int \ldots \int \prod_{l=1}^{l=N_1} d^3\mathbf{r_{1,l}} \prod_{k=1}^{k=N_2} d^3\mathbf{r_{2,k}}\ \exp\left(\eta_1\, u_{11} + \eta_2\, u_{22} + \sqrt{\eta_1 \eta_2}\, u_{12}\right) ,
$$

and

$$
\begin{aligned}
u_{11} &\equiv \frac{1}{N_1} \sum_{1 \leq i < j \leq N_1} \frac{1}{|\mathbf{r_{1,i}} - \mathbf{r_{1,j}}|} \quad , \quad u_{22} \equiv \frac{1}{N_2} \sum_{1 \leq i < j \leq N_2} \frac{1}{|\mathbf{r_{2,i}} - \mathbf{r_{2,j}}|} \quad , \\
u_{12} &\equiv \frac{1}{\sqrt{N_1 N_2}} \sum_{1 \leq i \leq N_1, 1 \leq j \leq N_2} \frac{1}{|\mathbf{r_{1,i}} - \mathbf{r_{2,j}}|} \quad .
\end{aligned}
$$

2.2 Mean field theory

We approximate the function Z_{int} for a large number of particles ($N_1 \gg 1$ and $N_2 \gg 1$) generalizing the approach of ref.[4, 5] for two kinds of particles. The function Z_{int} is expressed as a functional integral over all configurations with particle densities $\rho_1(\mathbf{x})$ and

$\rho_2(\mathbf{x})$ [$\rho_1(\mathbf{x})$ stands for the density of the particles of mass m_1 and $\rho_2(\mathbf{x})$ stands for the density of the particles of mass m_2].

$$Z_{int} = \int D\rho_1(.) \, D\rho_2(.) \, \mathrm{d}b_1 \, \mathrm{d}b_2 \, e^{-\frac{F-F_0}{T}} \tag{4}$$

with,

$$
\begin{aligned}
\frac{F - F_0}{T} &= -N_1 \frac{\eta_1}{2} \int \frac{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \rho_1(\mathbf{x})\,\rho_1(\mathbf{y}) - N_2 \frac{\eta_2}{2} \int \frac{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \rho_2(\mathbf{x})\,\rho_2(\mathbf{y}) \\
&\quad -\sqrt{N_1 N_2}\,\sqrt{\eta_1\eta_2} \int \frac{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \rho_1(\mathbf{x})\,\rho_2(\mathbf{y}) + \\
&\quad +N_1 \int \mathrm{d}^3\mathbf{x}\,\rho_1(\mathbf{x}) \ln\rho_1(\mathbf{x}) + N_2 \int \mathrm{d}^3\mathbf{x}\,\rho_2(\mathbf{x}) \ln\rho_2(\mathbf{x}) + \\
&\quad +iN_1\,b_1[1 - \int \mathrm{d}^3\mathbf{x}\,\rho_1(\mathbf{x})] + iN_2\,b_2[1 - \int \mathrm{d}^3\mathbf{x}\,\rho_2(\mathbf{x})] \ .
\end{aligned}
\tag{5}
$$

b_1 and b_2 are Lagrange multiplier enforcing the normalization of the densities:

$$\int \mathrm{d}^3\mathbf{x}\,\rho_1(\mathbf{x}) = 1 \quad , \quad \int \mathrm{d}^3\mathbf{x}\,\rho_2(\mathbf{x}) = 1 \ . \tag{6}$$

F stands for the free energy of the gas for the pair of densities (ρ_1, ρ_2), while

$$F_0 = -N_1 T \ln\left[\frac{eV}{N_1}\left(\frac{m_1 T}{2\pi}\right)^{\frac{3}{2}}\right] - N_2 T \ln\left[\frac{eV}{N_2}\left(\frac{m_2 T}{2\pi}\right)^{\frac{3}{2}}\right]$$

is the free energy of the perfect gas with masses m_1 and m_2. One recognizes the gravitational energy in the first two lines of eq.(5), while the third line contains the entropy.

Since the free energy becomes large in the thermodynamic limit ($N_1, N_2 \gg 1$), the functional integral Z_{int} (4) is dominated by the minima of $\frac{F-F_0}{T}$. Extremizing the free energy with respect to the pair of densities (ρ_1, ρ_2) yields the saddle point equations

$$
\begin{aligned}
\ln\rho_1(\mathbf{x}) &= a_1 + \eta_1 \int \frac{\mathrm{d}^3\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}\rho_1(\mathbf{y}) + \mu\,\eta_2 \int \frac{\mathrm{d}^3\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}\rho_2(\mathbf{y}) \\
\ln\rho_2(\mathbf{x}) &= a_2 + \frac{1}{\mu}\,\eta_1 \int \frac{\mathrm{d}^3\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}\rho_1(\mathbf{y}) + \eta_2 \int \frac{\mathrm{d}^3\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}\rho_2(\mathbf{y}) \ .
\end{aligned}
\tag{7}
$$

where we used eq.(2). These equations define the mean field approach. We set $a_1 = -1 + ib_1$ and $a_2 = -1 + ib_2$ and we denote by μ the ratio of masses of the two kinds of particles,

$$\mu \equiv \frac{m_1}{m_2} \ .$$

We set,

$$\rho_1(\mathbf{x}) = \exp[\Phi_1(\mathbf{x})] \quad , \quad \rho_2(\mathbf{x}) = \exp[\Phi_2(\mathbf{x})] \ . \tag{8}$$

Eqs.(7) give for the gravitational potential

$$U(\mathbf{x}) = -\frac{T}{m_1}[\Phi_1(\mathbf{x}) - a_1] = -\frac{T}{m_2}[\Phi_2(\mathbf{x}) - a_2] \tag{9}$$

created by the matter densities $\rho_1(.)$ and $\rho_2(.)$ at the point $\mathbf{x}$. Using eqs.(8)-(9), we see that these densities obey the Boltzmann laws

$$\rho_1(\mathbf{x}) = e^{a_1} \, e^{-\frac{m_1}{T}U(\mathbf{x})} \quad , \quad \rho_2(\mathbf{x}) = e^{a_2} \, e^{-\frac{m_2}{T}U(\mathbf{x})} \, , \tag{10}$$

containing the energy of a particle in this mean field gravitational potential, as it must be. e^{a_1} and e^{a_2} play the role of normalization constants.

Applying the Laplace operator to the saddle point equations (7) we find the differential equations

$$\Delta\Phi_1(\mathbf{x}) + 4\pi \, \eta_1 \, e^{\Phi_1(\mathbf{x})} + 4\pi \, \mu \, \eta_2 \, e^{\Phi_2(\mathbf{x})} = 0$$
$$\Delta\Phi_2(\mathbf{x}) + 4\pi \, \frac{\eta_1}{\mu} \, e^{\Phi_1(\mathbf{x})} + 4\pi \, \eta_2 \, e^{\Phi_2(\mathbf{x})} = 0 \, . \tag{11}$$

These equations are scale covariant. If (Φ_1, Φ_2) is a pair of solutions of eqs.(11), then the pair $(\Phi_{1\lambda}, \Phi_{2\lambda})$ defined by

$$\Phi_{1\lambda}(\mathbf{x}) = \Phi_1(\lambda\mathbf{x}) + \ln\lambda^2 \quad , \quad \Phi_{2\lambda}(\mathbf{x}) = \Phi_2(\lambda\mathbf{x}) + \ln\lambda^2 \tag{12}$$

is also a solution of eq.(11). This property is due to the scale behaviour of Newton's potential. Using eq.(9), we reduce eqs.(11) to a single equation

$$\Delta\Phi_1(\mathbf{x}) + 4\pi \, \eta_1 \, e^{\Phi_1(\mathbf{x})} + 4\pi\mu \, \eta_2 \, e^{a_2 - \frac{a_1}{\mu}} \, e^{\frac{\Phi_1(\mathbf{x})}{\mu}} = 0 \, . \tag{13}$$

Using eqs.(9) and (10), eq.(13) becomes in dimensionless coordinates,

$$\Delta U(\mathbf{x}) = \frac{4\,\pi\,G}{L} \left[m_1 \, N_1 \, \rho_1(\mathbf{x}) + m_2 \, N_2 \, \rho_2(\mathbf{x}) \right] \, . \tag{14}$$

We show in Appendix B that this equation is the condition of hydrostatic equilibrium for a two-component fluid once the ideal gas equation of state is postulated.

Therefore, the mean field approximation is equivalent to the hydrostatic description in the gaseous phase provided the ideal gas equation of state is assumed in the latter approach.

Notice that local equations of state other than the ideal gas are often assumed in the context of self-gravitating fluids[3]. As stressed in ref.[4], one needs long range forces other than gravitational in order to obtain a non-ideal local equation of state in thermal equilibrium.

3   Spherically symmetric case

For spherically symmetric configurations the mean field equations become ordinary non-linear differential equations. We express here the various thermodynamic quantities in terms of the solution of a single ordinary differential equation. Such equation reduces to the well-known isothermal sphere[3] if all particles have identical mass.

3.1   Reduction of the equation of saddle point

We consider here the spherically symmetric case where the mean field equations (11) take the form

$$\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}R^2} + \frac{2}{R}\frac{\mathrm{d}\Phi_1}{\mathrm{d}R} + 4\pi \, \eta_1 \, e^{\Phi_1(R)} + 4\pi \, \mu \, \eta_2 \, e^{\Phi_2(R)} = 0 \tag{15}$$

$$\frac{\mathrm{d}^2\Phi_2}{\mathrm{d}R^2} + \frac{2}{R}\frac{\mathrm{d}\Phi_2}{\mathrm{d}R} + 4\pi \, \frac{\eta_1}{\mu} \, e^{\Phi_1(R)} + 4\pi \, \eta_2 \, e^{\Phi_2(R)} = 0$$

where we work in a unit sphere. Therefore the radial variable runs in the interval $0 \leq R \leq R_{max}$, $R_{max} = \left(\frac{3}{4\pi}\right)^{\frac{1}{3}}$. Using the scale covariance of the mean field eqs. (11) by the transformation (12), we can set

$$
\begin{aligned}
\Phi_1(R) &= \chi_1\left(\lambda \, \frac{R}{R_{max}}\right) + \ln\left(\frac{\lambda^2}{3\eta_1^R}\right) \\
\Phi_2(R) &= \chi_2\left(\lambda \, \frac{R}{R_{max}}\right) + \ln\left(\frac{\lambda^2}{3\eta_2^R}\right) \, .
\end{aligned} \tag{16}
$$

We use the new parameters $\eta_1^R = \eta_1/R_{max}$ and $\eta_2^R = \eta_2/R_{max}$ in the spherically symmetric case. Hence the mean field eqs.(15) are transformed into a reduced system of the form,

$$
\begin{aligned}
\chi_1''(\lambda) + \frac{2}{\lambda}\chi_1'(\lambda) + e^{\chi_1(\lambda)} + \mu \, e^{\chi_2(\lambda)} &= 0 \\
\chi_2''(\lambda) + \frac{2}{\lambda}\chi_2'(\lambda) + \frac{1}{\mu} \, e^{\chi_1(\lambda)} + e^{\chi_2(\lambda)} &= 0 \, .
\end{aligned} \tag{17}
$$

Let us find the boundary conditions of these equations. In order to have a regular solution at origin we impose

$$
\chi_1'(0) = 0 \quad , \quad \chi_2'(0) = 0 \, .
$$

The system (17) is invariant under the transformation

$$
\lambda \to \lambda \, e^{\alpha} \quad , \quad \chi_i \to \chi_i - 2\alpha \quad i = 1, 2 \, .
$$

Hence, we can choose

$$
\chi_1(0) = 0 \, . \tag{18}
$$

without loosing generality. As we see below, the remaining boundary condition on $\chi_2(0)$ is not independent from η_1 and η_2.

The densities of the two kinds of particles ρ_1 and ρ_2 are to be normalized according to eq.(6). We obtain using eqs.(8) and (16),

$$
\eta_1^R = \frac{1}{\lambda} \int_0^\lambda dx \, x^2 \, e^{\chi_1(x)} \quad , \quad \eta_2^R = \frac{1}{\lambda} \int_0^\lambda dx \, x^2 \, e^{\chi_2(x)} \, . \tag{19}
$$

Using the reduced mean field equations (17), it is straightforward to show from eq.(19) that

$$
\begin{aligned}
\eta_1^R + \mu \, \eta_2^R &= -\lambda \chi_1'(\lambda) \tag{20} \\
\frac{1}{\mu} \, \eta_1^R + \eta_2^R &= -\lambda \chi_2'(\lambda) \, .
\end{aligned}
$$

Hence

$$
\chi_2'(\lambda) = \frac{1}{\mu} \, \chi_1'(\lambda) \, . \tag{21}
$$

Recalling that U is the gravitational potential (9) we see using eqs.(16), that

$$
\frac{T}{m_1} \, \chi_1'(\lambda) = \frac{T}{m_2} \, \chi_2'(\lambda)
$$

is the gravitational field at the boundary of the sphere ($R = R_{max}$) consistently with eq.(21). From eq.(21) we introduce a new parameter

$$
C \equiv \chi_2(\lambda) - \frac{1}{\mu} \, \chi_1(\lambda) \tag{22}
$$

8

independent of λ and function only of the physical parameters η_1^R and η_2^R. Notice the boundary condition

$$\chi_2(0) = C \ . \tag{23}$$

The reduced mean field equations (17) become a single equation with the parameter C as coefficient

$$\chi_1''(\lambda) + \frac{2}{\lambda} \chi_1'(\lambda) + e^{\chi_1(\lambda)} + \mu \ e^C \ e^{\frac{1}{\mu} \chi_1(\lambda)} = 0 \tag{24}$$

and the boundary conditions $\chi_1(0) = 0$, $\chi_1'(0) = 0$.

Using eqs.(8) and (16) we can express the densities of the two kinds of particle in terms of the solution of eq.(24)

$$\rho_1(R) = \frac{\lambda^2}{3\eta_1^R} \ e^{\chi_1\left(\lambda \frac{R}{R_{max}}\right)} \quad , \quad \rho_2(R) = \frac{\lambda^2 \ e^C}{3\eta_2^R} \ e^{\frac{1}{\mu}\chi_1\left(\lambda \frac{R}{R_{max}}\right)} \ . \tag{25}$$

Both $\chi_1(\lambda)$ and $\chi_2(\lambda)$ are functions of C [see (22)-(24)]. Inserting $\chi_1(x)$ and $\chi_2(x)$ in eq.(19), we see that η_1^R and η_2^R become functions of λ and C. Then expressing λ and C as functions of η_1^R and η_2^R, both densities of particles in eq.(25) become functions of the radial variable R and of the physical parameters η_1^R and η_2^R.

We find asymptotically from eq.(24)

$$\chi_1(\lambda) \overset{\lambda \to \infty}{=} -2\,\mu \ \log \lambda + \mathcal{O}(1)$$

for $\mu \geq 1$. Notice that this asymptotic behaviour may apply for non-physical values of λ where the gas is actually collapsed.

We now compute the thermodynamic quantities as functions of the physical parameters η_1^R and η_2^R.

3.2   Free energy

Let us start by computing the free energy. Using eqs. (5)-(8) we find

$$\frac{F - F_0}{T} = \frac{N_1}{2} \left(a_1 + \int \mathrm{d}^3\mathbf{x} \ \Phi_1(\mathbf{x}) \ e^{\Phi_1(\mathbf{x})} \right) + \frac{N_2}{2} \left(a_2 + \int \mathrm{d}^3\mathbf{x} \ \Phi_2(\mathbf{x}) \ e^{\Phi_2(\mathbf{x})} \right) \ . \tag{26}$$

We express now the Lagrange multipliers a_1 and a_2 as functions of the physical parameters η_1^R and η_2^R. In the spherically symmetric case the integration over the angles in eqs.(7) yields

$$
\begin{aligned}
\Phi_1(R) &= a_1 + 4\pi\eta_1 \left(\frac{1}{R} \int_0^R \mathrm{d}R' \ R'^2 \ e^{\Phi_1(R')} + \int_R^{R_{max}} \mathrm{d}R' \ R' \ e^{\Phi_1(R')} \right) \\
&\quad + 4\pi \ \mu \ \eta_2 \left(\frac{1}{R} \int_0^R \mathrm{d}R' \ R'^2 \ e^{\Phi_2(R')} + \int_R^{R_{max}} \mathrm{d}R' \ R' \ e^{\Phi_2(R')} \right) \\
\Phi_2(R) &= a_2 + 4\pi \frac{\eta_1}{\mu} \left(\frac{1}{R} \int_0^R \mathrm{d}R' \ R'^2 \ e^{\Phi_1(R')} + \int_R^{R_{max}} \mathrm{d}R' \ R' \ e^{\Phi_1(R')} \right) \\
&\quad + 4\pi \ \eta_2 \left(\frac{1}{R} \int_0^R \mathrm{d}R' \ R'^2 \ e^{\Phi_2(R')} + \int_R^{R_{max}} \mathrm{d}R' \ R' \ e^{\Phi_2(R')} \right) \ .
\end{aligned}
\tag{27}
$$

We introduce the densities for the two kinds of particles at the boundary $(R = R_{max})$

$$f_1 \equiv e^{\Phi_1(R_{max})} = \frac{\lambda^2}{3\eta_1^R} \ e^{\chi_1(\lambda)} \quad , \quad f_2 \equiv e^{\Phi_2(R_{max})} = \frac{\lambda^2 \ e^C}{3 \ \eta_2^R} \ e^{\frac{\chi_1(\lambda)}{\mu}} \ , \tag{28}$$
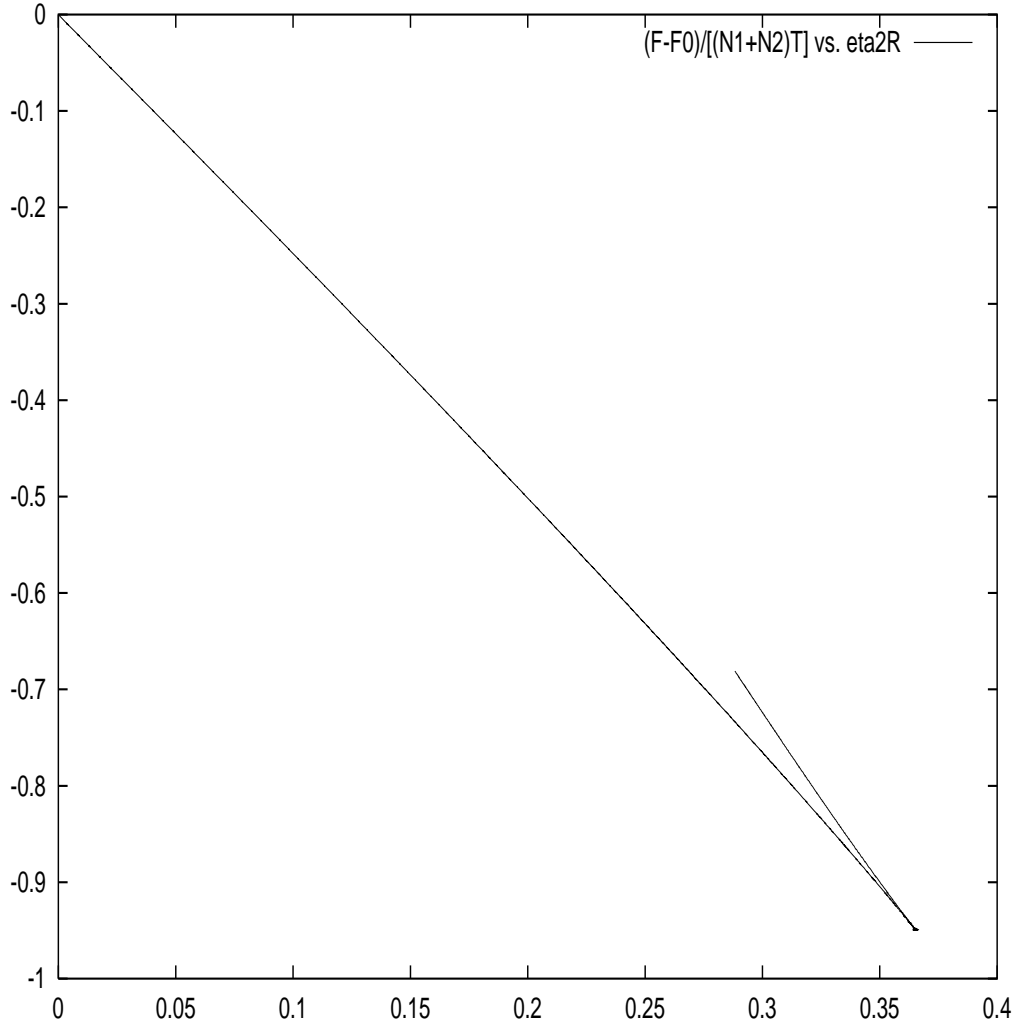
Figure 1: Free energy versus η_2^R for $\frac{m_1}{m_2} = 4$, $\frac{N_1}{N_2} = \frac{1}{3}$ and therefore $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$.
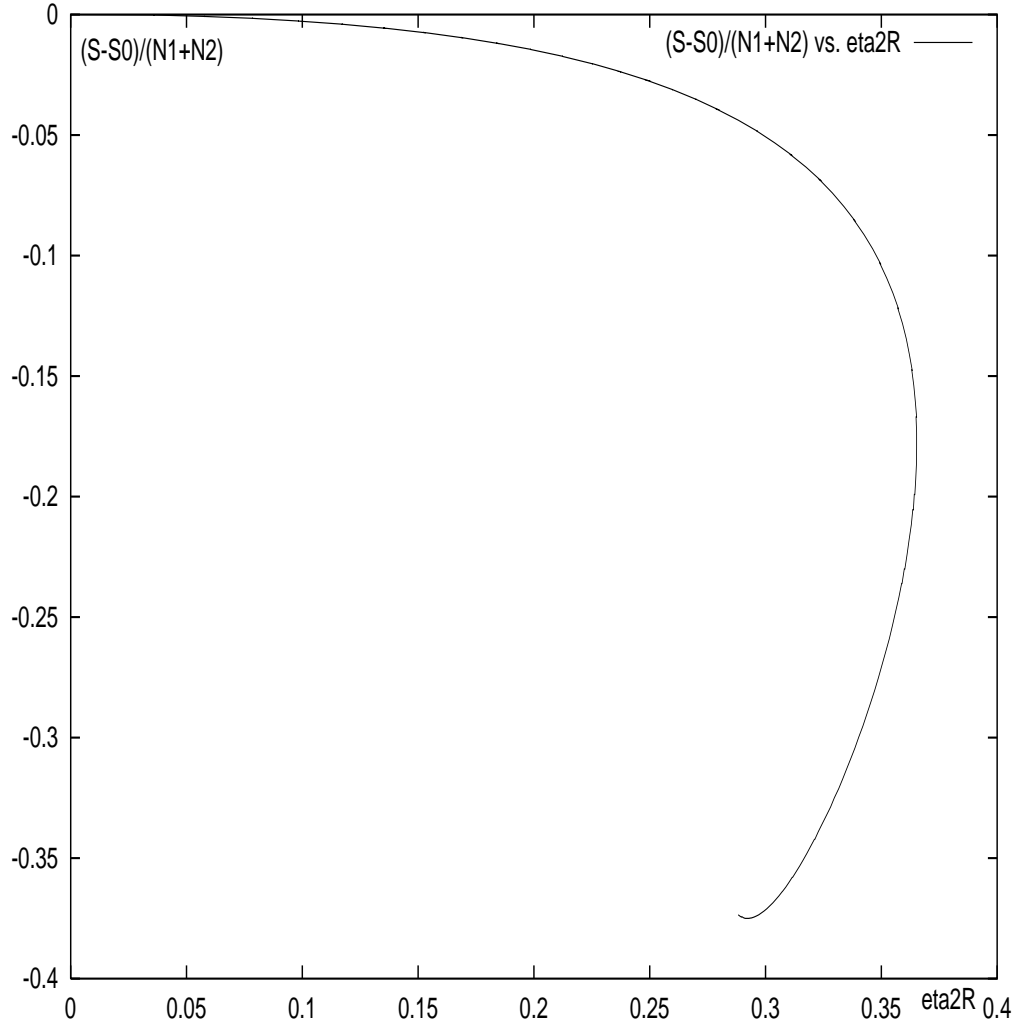
Figure 2: Entropy versus η_2^R for $\frac{m_1}{m_2} = 4$, $\frac{N_1}{N_2} = \frac{1}{3}$ and therefore $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$.

where we used eqs.(8) and (16). Notice that λ and C are functions of η_1^R and η_2^R as explained by the end of sec. 3.1. Hence, f_1 and f_2 are functions of η_1^R and η_2^R.

We find for $R = R_{max}$ the following expressions for the Lagrange multipliers using the normalization of the densities (6)

$$a_1 = \ln f_1 - \eta_1^R - \mu\, \eta_2^R \quad , \quad a_2 = \ln f_2 - \frac{1}{\mu}\, \eta_1^R - \eta_2^R \ . \tag{29}$$

Inserting these expressions (29) into the free energy (26), we find

$$\begin{aligned}
\frac{F - F_0}{T} &= \frac{N_1}{2}\left[\ln f_1 - \eta_1^R - \mu\, \eta_2^R\right] + \frac{N_2}{2}\left[\ln f_2 - \frac{1}{\mu}\, \eta_1^R - \eta_2^R\right] \\
&\quad + \frac{N_1}{2}\int d^3\mathbf{x}\, \Phi_1(\mathbf{x})\, e^{\Phi_1(\mathbf{x})} + \frac{N_2}{2}\int d^3\mathbf{x}\, \Phi_2(\mathbf{x})\, e^{\Phi_2(\mathbf{x})} \ .
\end{aligned} \tag{30}$$

We compute the integrals in the second line in appendix A and we find

$$\frac{F - F_0}{T} = N_1[\ln f_1 - \eta_1^R - \mu\, \eta_2^R + 3(1 - f_1)] + N_2[\ln f_2 - \frac{1}{\mu}\, \eta_1^R - \eta_2^R + 3(1 - f_2)] \ .$$

The free energy as well as the other physical quantities are functions of η_1^R and η_2^R. The parameters η_1^R and η_2^R are linked by the relation,

$$\eta_2^R = \frac{m_2^2\, N_2}{m_1^2\, N_1}\, \eta_1^R \ .$$

Hence, for fixed ratios $\frac{N_1}{N_2}$ and $\frac{m_2}{m_1}$, the physical quantities depend only on η_1^R or on η_2^R. In that case it is simpler to see the physical quantities as functions of η_2^R or η_1^R on a two dimensional plot than to watch the three dimensional surfaces for the physical quantities as functions of η_2^R **and** η_1^R.

We plot in fig.1 the free energy as a function of η_2^R for fixed $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$. In the limit where the particles of mass m_1 dominate ($N_1 \gg N_2$) the free energy becomes

$$\frac{F}{T} \overset{N_1 \gg N_2}{=} N_1[\ln f_1 - \eta_1^R + 3(1 - f_1)] \ .$$

We recognize here the free energy of a self-gravitating gas with N_1 particles of mass m_1 (see ref.[4]).

3.3   Energy

We compute here the gravitational energy of the gas. The density of gravitational energy is

$$\epsilon_P(\mathbf{x}) = \frac{1}{2}\left(\frac{m_1 N_1 \rho_1(\mathbf{x})}{V} + \frac{m_2 N_2 \rho_2(\mathbf{x})}{V}\right) U(\mathbf{x})$$

where U is the gravitational potential (9). Hence,

$$\epsilon_P(\mathbf{x}) = -\frac{N_1 T}{2V}\, e^{\Phi_1(\mathbf{x})}\, [\Phi_1(\mathbf{x}) - a_1] - \frac{N_2 T}{2V}\, e^{\Phi_2(\mathbf{x})}\, [\Phi_2(\mathbf{x}) - a_2] \ . \tag{31}$$

12

Using the expressions for the Lagrange multipliers (29), we obtain for the gravitational energy density in the spherically symmetric case

$$
\epsilon_P(\mathbf{x}) = \frac{N_1 T}{2V}\, \rho_1(R) \left[\ln\left(\frac{\rho_1(R_{max})}{\rho_1(R)} \right) - \eta_1^R - \mu\eta_2^R \right]
$$
$$
+ \frac{N_2 T}{2V}\, \rho_2(R) \left[\ln\left(\frac{\rho_2(R_{max})}{\rho_2(R)} \right) - \frac{1}{\mu}\eta_1^R - \eta_2^R \right] . \tag{32}
$$

Integrating the energy density (31), with the help of eqs. (6) and (29) we obtain

$$
E_P = \frac{N_1 T}{2} \left[\ln f_1 - \eta_1^R - \mu\, \eta_2^R \right] + \frac{N_2 T}{2} \left[\ln f_2 - \frac{1}{\mu}\, \eta_1^R - \eta_2^R \right]
$$
$$
- \frac{N_1 T}{2} \int d^3\mathbf{x}\, \Phi_1(\mathbf{x})\, e^{\Phi_1(\mathbf{x})} - \frac{N_2 T}{2} \int d^3\mathbf{x}\, \Phi_2(\mathbf{x})\, e^{\Phi_2(\mathbf{x})} . \tag{33}
$$

The integrals in the second line are computed in appendix A yielding for the gravitational energy

$$
E_P = 3T\, [N_1(f_1 - 1) + N_2(f_2 - 1)] . \tag{34}
$$

3.4 Entropy

Using the gravitational energy (34) and the free energy (30) we obtain for the entropy

$$
S = S_0 + N_1[6(f_1 - 1) - \ln f_1 + \eta_1^R + \mu\, \eta_2^R] + N_2[6(f_2 - 1) - \ln f_2 + \frac{1}{\mu}\, \eta_1^R + \eta_2^R]
$$

where S_0 is the entropy of the perfect gas.

$$
S_0 = N_1 \left(\ln\left[\frac{V}{N_1} \left(\frac{m_1 T}{2\pi} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right) + N_2 \left(\ln\left[\frac{V}{N_2} \left(\frac{m_2 T}{2\pi} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right) .
$$

We plot in Fig. 2 the entropy S as a function of η_2^R for $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$.

3.5 Local pressure

Since the system is non-homogeneous the local pressure is not uniform. The density of gravitational force is

$$
\mathbf{F}(\mathbf{r}) = -\frac{m_1\, N_1\, \rho_1(\mathbf{r}) + m_2\, N_2\, \rho_2(\mathbf{r})}{V}\, \mathbf{grad}[U(\mathbf{r})]
$$

where $U(\mathbf{r})$ is the gravitational potential (9).
We obtain for the force at the point $\mathbf{r}$ using eqs.(8) and (9),

$$
\mathbf{F}(\mathbf{r}) = \frac{N_1 T}{V}\, \mathbf{grad}(e^{\Phi_1(\mathbf{r})}) + \frac{N_2 T}{V}\, \mathbf{grad}(e^{\Phi_2(\mathbf{r})}) .
$$

The link between the density of force and the pressure is

$$
\mathbf{F}(\mathbf{r}) = \mathbf{grad}[P(\mathbf{r})] .
$$

Using eq.(8), the local pressure is given by,

$$
P(\mathbf{r}) = \frac{N_1 T}{V}\, \rho_1(\mathbf{r}) + \frac{N_2 T}{V}\, \rho_2(\mathbf{r}) . \tag{35}
$$

13

This is the **local** equation of state for the self-gravitating gas with two kinds of particles. We see that it locally coincides with the equation of state of a perfect gas. Since the gas is inhomogeneous, the pressure acting on any finite volume will **not** obey the ideal equation of state.

For a point $\mathbf{r}$ at the boundary, using eqs.(8), (28) and (35) yields the external pressure as a function of η_1^R and η_2^R as,

$$P = \frac{N_1 T}{V} f_1 + \frac{N_2 T}{V} f_2 \; , \tag{36}$$

where f_1 and f_2 are defined by eq.(28). This is the equation of state of the gas as a whole that we plot in fig.3.

Combining eqs.(34) and (36) yields the virial theorem

$$\frac{PV}{T} = N_1 + N_2 + \frac{E_P}{3T} \; .$$

3.6   Physical behaviour of the system

This self-gravitating system formed by two kinds of particles can be in two phases: gaseous and condensed. The first one corresponds to η_1 and η_2 between the origin and their collapse values. In the gaseous phase the free energy has a minimum for the pair of densities (ρ_1, ρ_2), solutions of the saddle point equations (7). This pair of densities is the most probable distribution which become absolutely certain in the thermodynamic limit. All thermodynamic quantities follows from this pair of densities (ρ_1, ρ_2).

In the condensed phase, (ρ_1, ρ_2) from mean field does not describe the particle distribution and the mean field approach fails to describe the condensed phase. It may be studied by Monte-Carlo methods as in ref.[4].

When the physical parameters $\eta_1^R = \eta_2^R = 0$ we retrieve the perfect gas. When η_1^R and η_2^R increase, the gas becomes denser at the center of the sphere $(R = 0)$ and less dense at the boundary $(R = R_{max})$ because of gravitational attraction [see Figs. 7-8]. This effect is more acute for the heavier particles showing that more massive particles diffuse to the denser regions.

The equation of state is depicted in fig.3. In the ideal gas limit, $\eta_1^R = \eta_2^R = 0$, $PV = (N_1 + N_2)T$. In the case $\eta_1^R = 0$ (gas of particles of mass m_2) and in the case $\eta_2^R = 0$ (gas of particles of mass m_1) we recover the equation of state of the self-gravitating gas with one kind of particles [4].

We call critical values η_1^{RC} and η_2^{RC} the points where $PV/[(N_1 + N_2)T]$ exhibits a vertical slope. η_1^{RC} and η_2^{RC} define a critical line in the (η_1^R, η_2^R) plane. Namely, for each value of N_1/N_2 we have a different pair of critical points η_1^{RC} and η_2^{RC}. We plot the critical line in the (η_1^R, η_2^R) plane in fig.4.

The surface pressure has a rim on the canonical critical line [see fig. 3]. The projection of this rim on the (η_1^R, η_2^R) plane yields the critical line plotted in fig.4. For a fixed N_1/N_2 we get a section of the equation of state surface depicted in fig.5. This section turns to have a form analogous to the equation of state for the self-gravitating gas with one kind of particles [4].

By analogy with the gas with one kind of particles, we expect that the gas collapses in the condensed phase for values of η_1^R and η_2^R slightly below η_1^{RC} and η_2^{RC} where the saddle point approximation breaks down.

PV/[(N1+N2)T] vs. eta1R and eta2R

+

PV/[(N1+N2)T]

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2

0

0.5

1

1.5

2

2.5

3

0

0.5

1
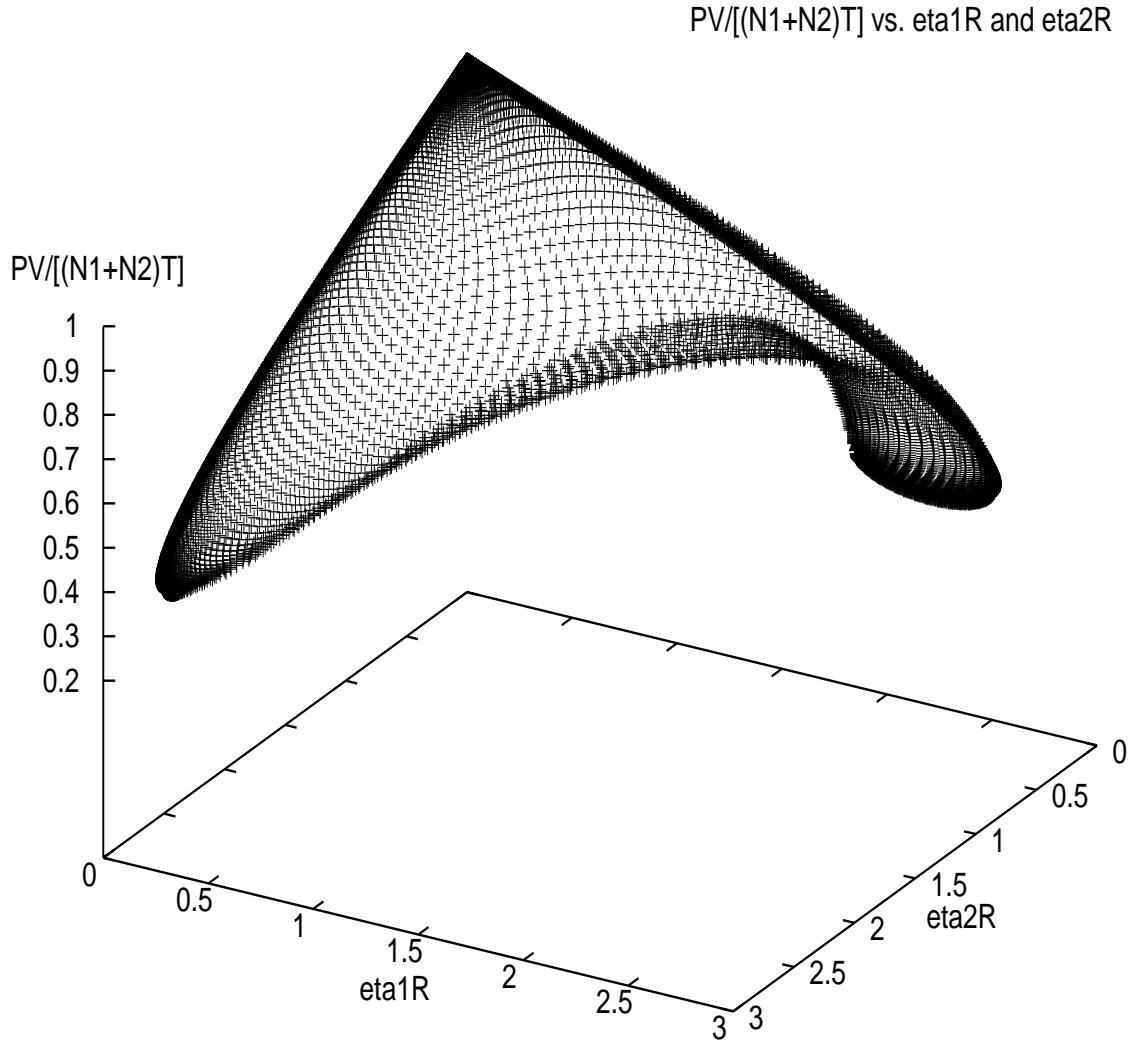
1.5

2

2.5

3

eta1R

eta2R

Figure 3: Equation of state: $\frac{PV}{(N_1+N_2)T}$ versus η_1^R and η_2^R. We choose $\frac{m_1}{m_2} = 4$.
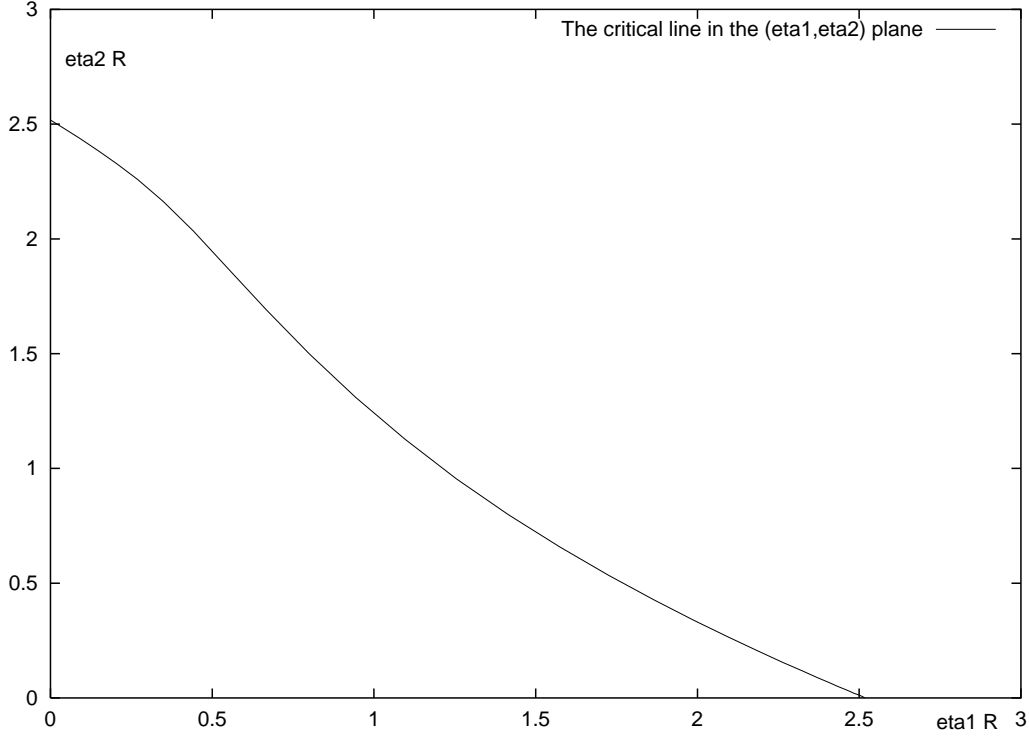
Figure 4: The critical line in the (η_1^R, η_2^R) plane. The mean field approximation is valid in the region below the critical line

The physical quantities exhibit a square root Riemann sheet structure as functions of η_1^R and η_2^R. The branch points are on the critical line for the canonical ensemble. The lower branch [see Figs. 3 and 5 for the equation of state and figs. 7-8 for the densities] describe a phase absent in the canonical ensemble. Such phase is realized and is stable in the microcanonical ensemble as it was the case for the gas with one kind of particles[4].

We plot in Fig.6 the value of $\eta_1^R + \mu\,\eta_2^R$ at the critical line as a function of the number of particles N_1/N_2. This quantity is proportional to the total mass of the gas $m_1\,N_1 + m_2\,N_2$. This critical parameter $\eta_1^{RC} + \mu\,\eta_2^{RC}$ interpolates between the two limiting cases ($N_1 \gg N_2$ and $N_1 \ll N_2$) where one kind of particles dominate over the others. When $N_1 \gg N_2$ the particles of mass m_1 dominate and $\eta_1^{RC} \gg \eta_2^{RC}$. In this limiting case, $\eta_1^{RC} + \mu\,\eta_2^{RC} \to \eta^{RC} = 2.518\ldots$ which is the critical value for the self gravitating gas with one kind of particles associated to the Jeans instability[3, 4]. When $N_2 \gg N_1$ the particles of mass m_2 dominate and $\eta_2^{RC} \gg \eta_1^{RC}$. In this limiting case, $\eta_1^{RC} + \mu\,\eta_2^{RC} \to \mu\,\eta^{RC} = 10.04\ldots$ for the mass ratio $\mu = 4$ corresponding to a mixture of hydrogen and helium.

From eq.(35) we see that the partial pressures of the particles with mass m_1 and m_2 are given by,

$$P_1(\mathbf{r}) = \frac{N_1 T}{V}\,e^{\Phi_1(\mathbf{r})} \quad , \quad P_2(\mathbf{r}) = \frac{N_2 T}{V}\,e^{\Phi_2(\mathbf{r})} \ . \tag{37}$$

The pressure contrast is defined by the ratio of the pressure at the center and the pressure at the boundary: $P(0)/P(R_{max})$ [3]. We find from eqs.(18), (23), (25) and (35)

$$\alpha \equiv \frac{P(0)}{P(R_{max})} = \frac{1 + \mu^2\,e^C}{e^{\chi_1(\lambda)} + \mu^2\,e^{C+\chi_1(\lambda)/\mu}} \ .$$

We extend this definition to each kind of particles and say that the partial contrasts are given by $P_1(0)/P_1(R_{max})$ for the particles of mass m_1 and $P_2(0)/P_2(R_{max})$ for the particles
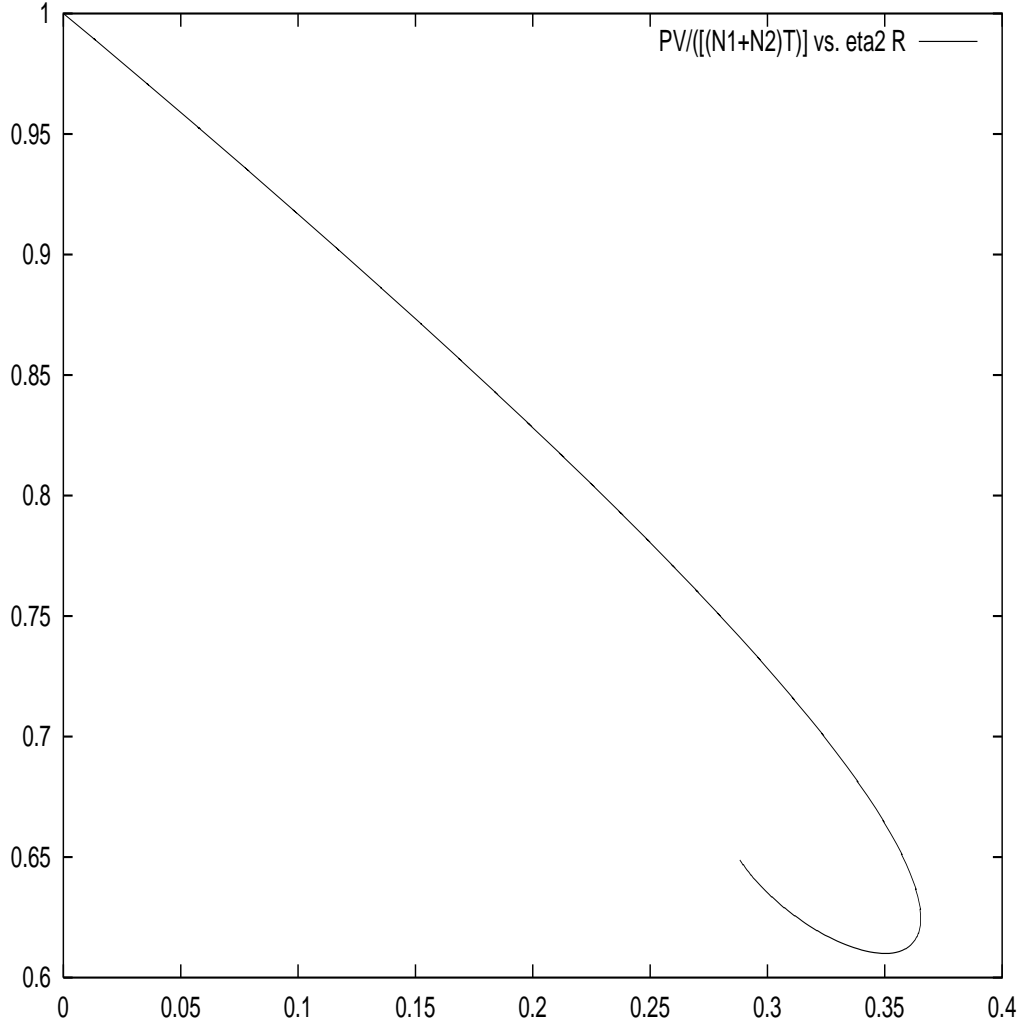
16

Figure 5: Equation of state $\frac{PV}{(N_1+N_2)T}$ as a function of η_2^R for $\frac{m_1}{m_2} = 4$, $\frac{N_1}{N_2} = \frac{1}{3}$ and therefore $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$.
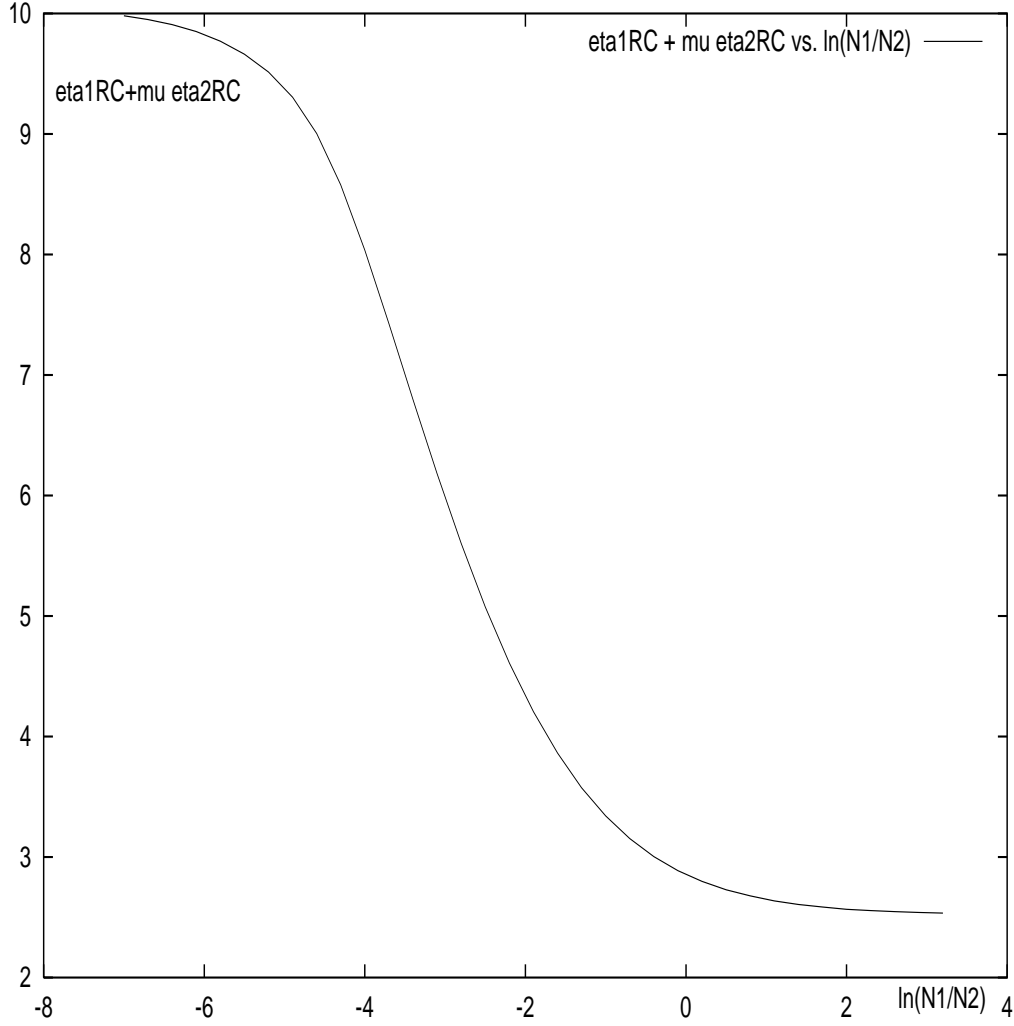
Figure 6: Critical value of the physical parameter $\eta_1^{RC} + \mu\,\eta_2^R$ versus $\ln\frac{N_1}{N_2}$ for $\frac{m_1}{m_2} = 4$.

of mass m_2. We find from eqs.(18), (23), (25) and (37),

$$\alpha_1 \equiv \frac{P_1(0)}{P_1(R_{max})} = e^{-\chi_1(\lambda)} \quad \text{and} \quad \alpha_2 \equiv \frac{P_2(0)}{P_2(R_{max})} = e^{-\chi_1(\lambda)/\mu} \ . \tag{38}$$

We plot the contrast and the partial contrasts in fig. 9. We see that the contrast α takes here **lower** values than for a gas with one kind of particles. On the contrary, α_1, the partial contrast for the **heavier** particles, takes **higher values** than the contrast for a gas with one kind of particles. This is due to the fact that the overdensity of particles in the center is more acute for the heavier ones as noticed above.

The particle density and pressure has its maximum at the origin and its minimum at the boundary, as expected. However, their ratio (contrast) is much **larger** for the heavier particles than for the lighter ones. They are related through [see eq.(38)],

$$\alpha_1 = [\alpha_2]^\mu$$

where $\mu = m_1/m_2$. Since $\alpha_2 > 1$, when $m_1 > m_2$, we can get $\alpha_1 \gg \alpha_2$ [see fig.9]. In particular, α_1 may be much larger than the contrast in the gas with one kind of particles[3, 4]. In summary, this shows that the heavier particles diffuse to the denser regions.
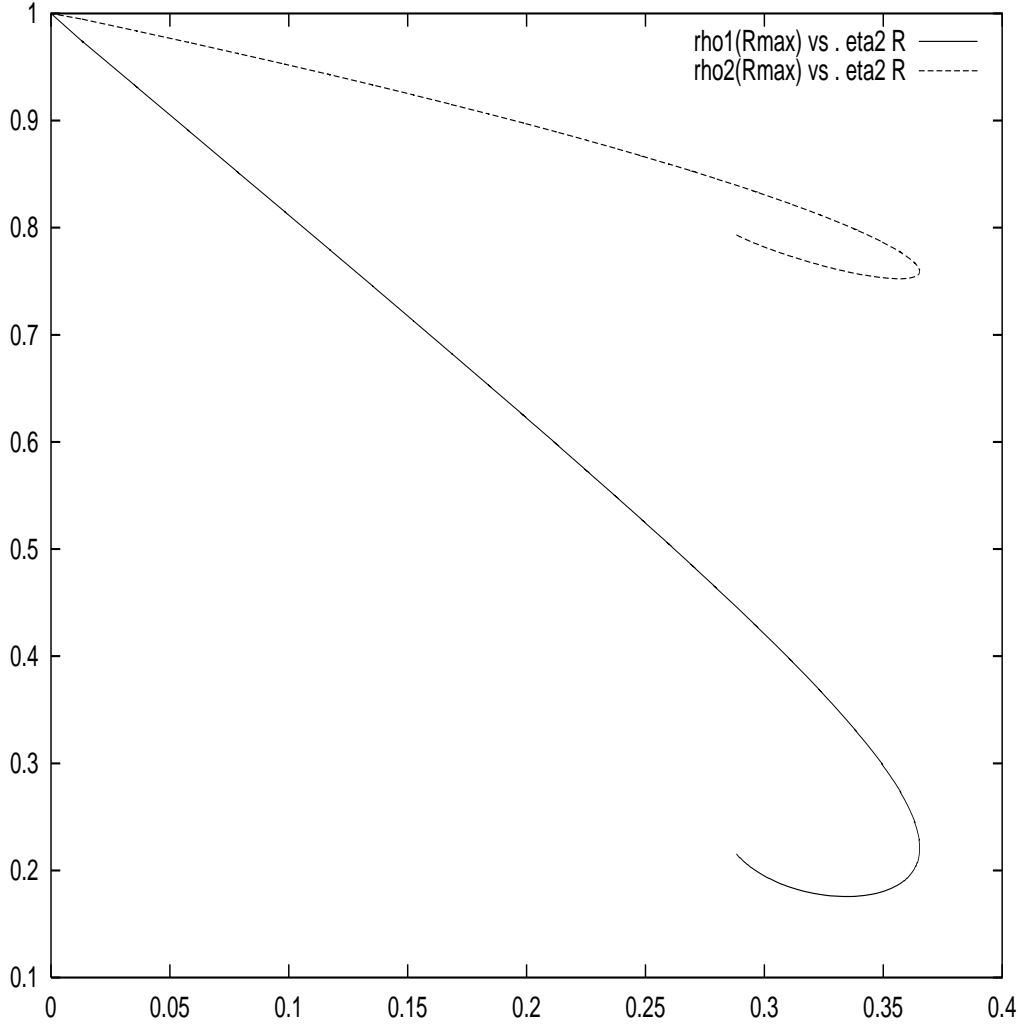
Figure 7: Density of particles of mass m_1 $[\rho_1(R_{max})]$ and density of particles of mass m_2 $[\rho_2(R_{max})]$ at the boundary versus η_2^R where the mass ratio is $\frac{m_1}{m_2} = 4$, the number of particles ratio is $\frac{N_1}{N_2} = \frac{1}{3}$ and then $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$.

3.7   Scaling law

We compute here the mass $M(R)$ inside a sphere of radius R $(0 \leq R \leq R_{max})$.

Using Gauss's theorem and recalling that U is the gravitational potential (9) we find that

$$M(R) = -\frac{m_1 N_1}{\eta_1} R^2 \, \Phi_1'(R) \ .$$

Using eqs.(16) we obtain

$$M(R) = -\frac{m_1 N_1 \lambda}{\eta_1^R} \left(\frac{R}{R_{max}}\right)^2 \chi_1'\left(\lambda \frac{R}{R_{max}}\right) \ .$$

As for the self-gravitating gas where all particles have the same mass, the mass $M(R)$ for the self-gravitating gas with two kinds of particles follows approximately the scaling law

$$M(R) \approx \mathcal{C} \, R^d \ .$$

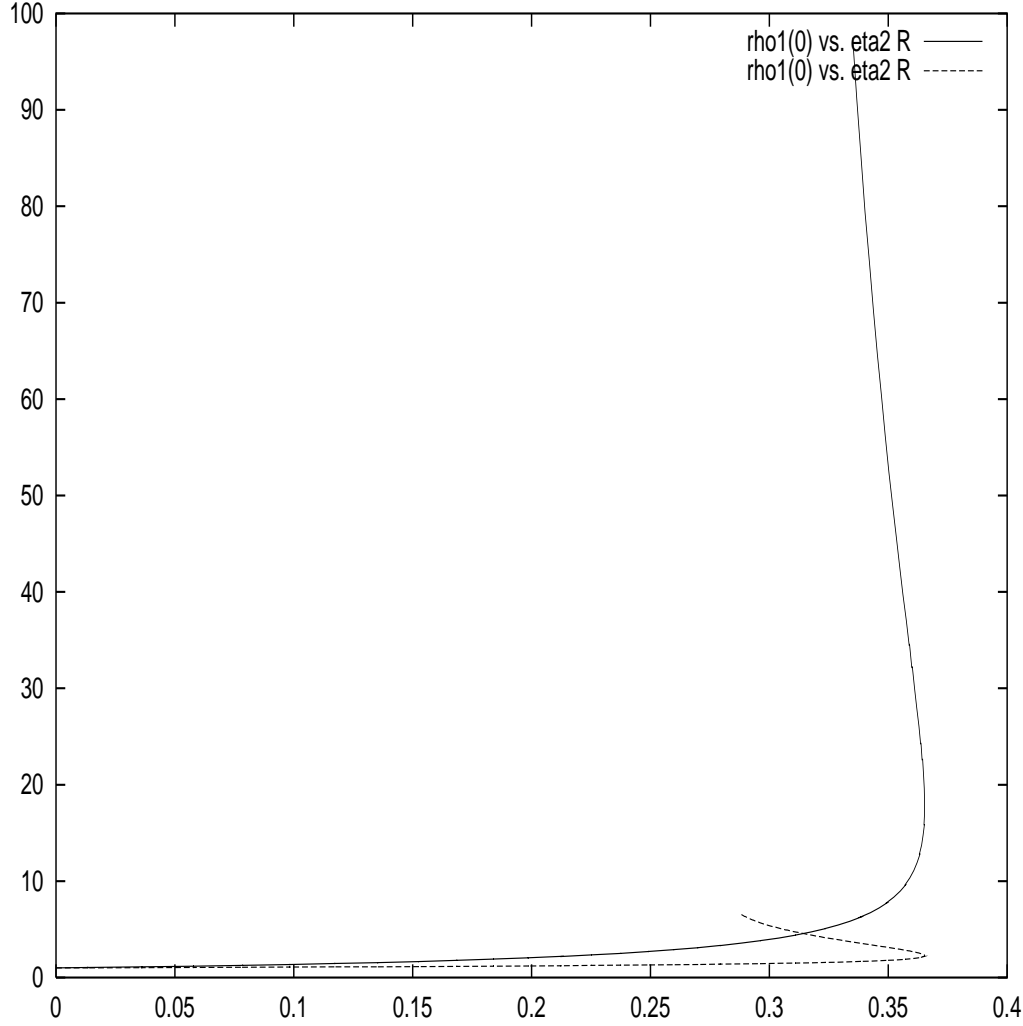This indicates a fractal distribution with Haussdorf dimension d.

Figure 8: Density of particles of mass m_1 [$\rho_1(0)$] and density of particles of mass $m_2(0)$ [$\rho_2(0)$] at the origin versus η_2^R where the mass ratio is $\frac{m_1}{m_2} = 4$, the number of particles ratio is $\frac{N_1}{N_2} = \frac{1}{3}$ and then $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$.

d decreases with η_1^R and η_2^R from the value $d = 3$ for the ideal homogeneous gas ($\eta_1^R = \eta_2^R = 0$) till $d \approx 1.6$ in the canonical critical line. The Haussdorf dimension keeps decreasing beyond the canonical critical line in the stable phase of the microcanonical ensemble.

We plot in fig.10 the mass $M(R)$ for $M(R)$ greater than 10% of the total mass of the gas for several values of η_1^R choosing the ratio η_1^R/η_2^R to be $0.534\ldots$. We exclude the region $M(R) < 0.1$ where the mass distribution is almost uniform. This local uniformity is simply explained by the absence of gravitational forces at the origin $R = 0$ due to the spherical symmetry.

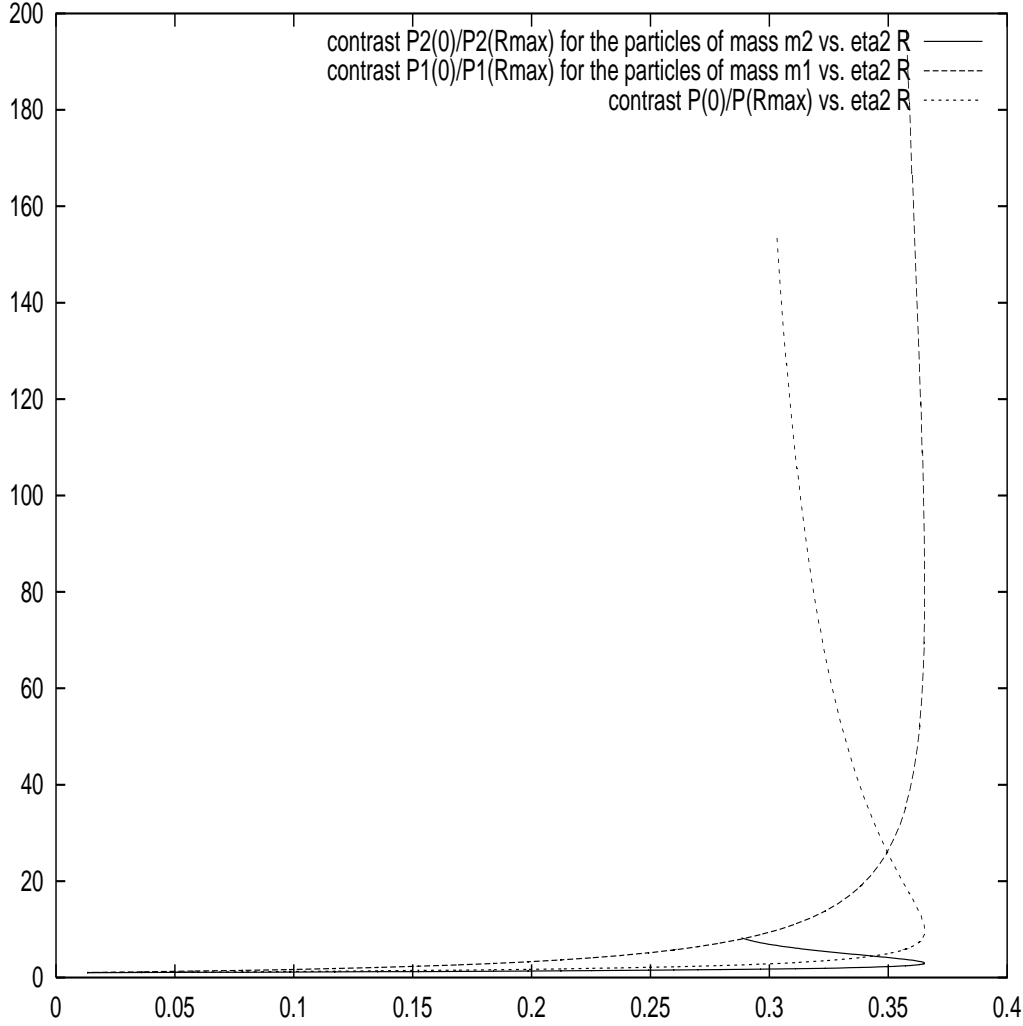| η_1^R	η_2^R	d	C
0.01	0.0187	2.99	1.00
0.3	0.562	2.72	1.03
0.6	1.12	2.31	1.07
$\eta_1^{RC} = 0.80...$	$\eta_2^{RC} = 1.49...$	1.66	1.03

Figure 9: Contrast $P(0)/P(R_{max})$, partial contrast $P_1(0)/P_1(R_{max})$ for the particles of mass m_1 and partial contrast $P_2(0)/P_2(R_{max})$ for the particles of mass m_2 versus η_2^R where the mass ratio is $\frac{m_1}{m_2} = 4$, the number of particles ratio is $\frac{N_1}{N_2} = \frac{1}{3}$ and then $\frac{\eta_1^R}{\eta_2^R} = \frac{16}{3}$. We have at the critical point $P_1(0)/P_1(R_{max}) \simeq 81. > P(0)/P(R_{max}) \simeq 9.9 > P_2(0)/P_2(R_{max}) \simeq 3.0$.

TABLE 1. The fractal dimension d and the proportionality coefficient $\mathcal{C}$ from a fit according to $M(R) \approx \mathcal{C} \ R^d$ for $\frac{m_1}{m_2} = 4$, $\frac{N_1}{N_2} = 0.0334$ and then $\frac{\eta_1^R}{\eta_2^R} = 0.534$ and different values of η_1^R and η_2^R.

As shown in fig. 11, the Haussdorf dimension at the canonical critical line turns to be **independent** of the ratio η_1^R/η_2^R. d coincides within the numerical precision with the Haussdorf dimension $d \approx 1.6$ at the canonical critical point of the gas with one kind of particles [4]. This indicates that the Haussdorf dimension at the canonical critical line is an universal value, independent of the gas composition.
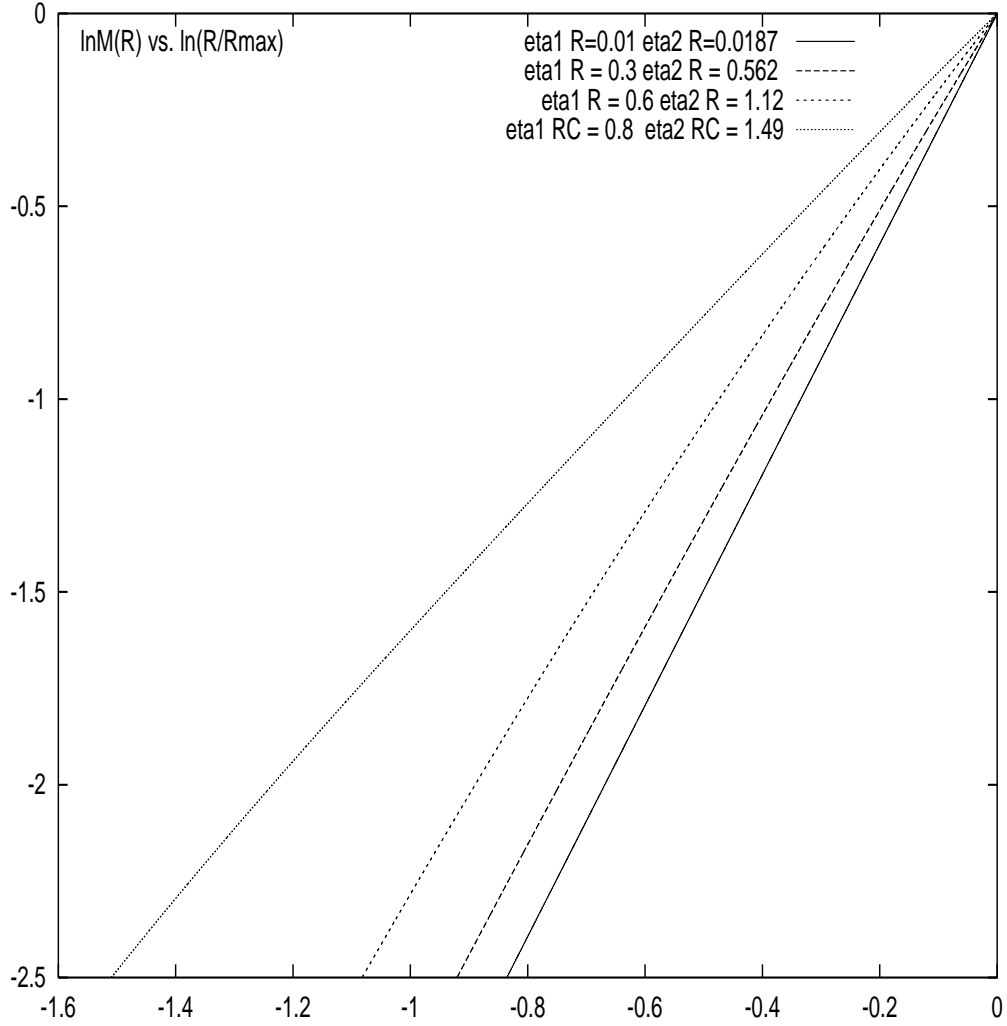
Figure 10: $\ln M(R)$ versus $\ln \frac{R}{R_{max}}$ for $\frac{m_1}{m_2} = 4$, $\frac{N_1}{N_2} = 0.0334\ldots$ and then $\frac{\eta_1^R}{\eta_2^R} = 0.534\ldots$ for different values of η_1^R : $\eta_1^R = 0.01$, $\eta_1^R = 0.03$, $\eta_1^R = 0.06$ and the canonical critical point $\eta_1^{RC} = 0.8002\ldots$

η_1^{RC}	η_2^{RC}	d	C
0.03627	0.0187	1.62	1.19
0.6682	1.688	1.61	1.04
1.096	1.126	1.65	1.03
1.415	0.7978	1.62	1.04

TABLE 2. The fractal dimension d and the proportionality coefficient C from a fit according to $M(R) \approx C \, R^d$ on the critical canonical line for different values of η_1^{RC} and η_2^{RC}

3.8   Critical behaviour of the thermodynamic functions

According to the behaviour of the free energy near the critical line (see Fig. 1), we find that the first derivatives of the free energy (energy, pressure) are continuous, while the second derivatives (specific heats, compressibility) are discontinuous. Using eq.(34) and the form of the functions f_1 and f_2 near the critical line (see Fig. 7), we find that the
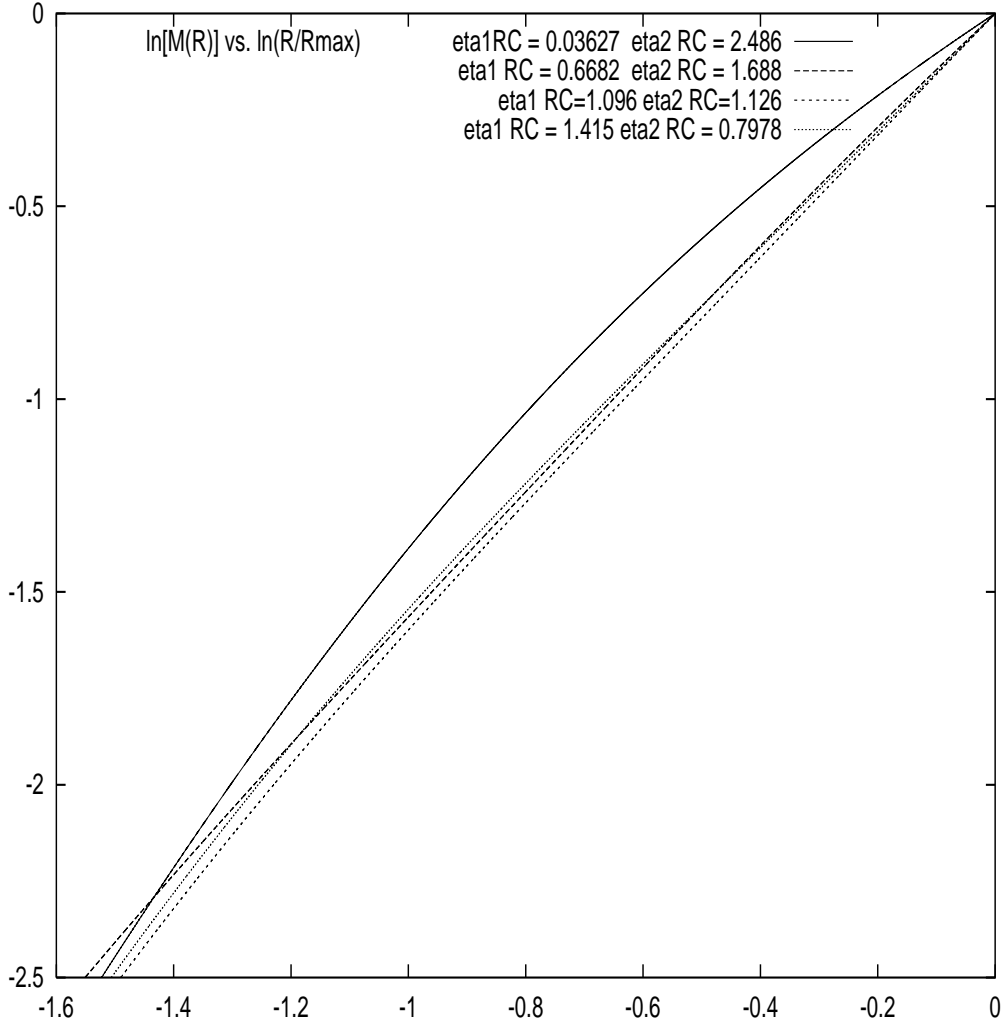
Figure 11: $\ln M(R)$ versus $\ln \frac{R}{R_{max}}$ for $\frac{m_1}{m_2} = 4$ on the critical canonical line for different values of η_1^R and η_2^R: $\eta_1^{RC} = 0.03627\ldots$, $\eta_2^{RC} = 2.486\ldots$; $\eta_1^{RC} = 0.6682\ldots$, $\eta_2^{RC} = 1.688\ldots$; $\eta_1^{RC} = 1.096\ldots$, $\eta_2^{RC} = 1.126\ldots$ and $\eta_1^{RC} = 1.415\ldots$, $\eta_2^{RC} = 0.7978\ldots$ The Haussdorf dimension d turns to be independent of the composition of the gas taking the value $d \approx 1.6$. It coincides within the numerical accuracy with the Haussdorf dimension for the gas with one kind of particles at the canonical critical point[4].

energy has two branches E_+ and E_- which behave near the critical point as,

$$E_\pm = \frac{3}{2}(N_1 + N_2)T \pm DT\sqrt{\eta_2^{RC} - \eta_2^R} \tag{39}$$

where D is a positive constant. Deriving the energy (39) with respect to T and recalling eq.(2) for η_2, we obtain the two branches of the specific heat at constant volume

$$C_{v\pm} = \frac{3}{2}(N_1 + N_2) \pm \frac{DT}{2\sqrt{\eta_2^{RC} - \eta_2^R}} \ .$$

Here, E_+ and C_{v+} stand for the gaseous stable phase in the canonical ensemble and E_- and C_{v-} correspond to a phase only realized in the microcanonical ensemble [4]. We see that $E_+ = E_-$ at the critical point and hence E is continuous at criticality while C_v exhibits there an infinite discontinuity. (As is clear, negative values of C_v cannot be realized in the canonical ensemble[6]).

4   A self-gravitating gas with n kinds of particles

The generalization of the treatment given in previous sections to a self-gravitating gas with n kinds of particles is straightforward. We give below the mean field equations and the more relevant results.

The relevant parameters of the gas are now

$$\eta_i = \frac{G \, m_i^2 \, N_i}{LT} \quad , \quad 1 \leq i \leq n,$$

where N_i is the number of particles of mass m_i. We assume that N_i/L stay fixed while $L \rightarrow \infty$, $N_i \rightarrow \infty, 1 \leq i \leq n$. Therefore, the ratios N_i/N_j also stay fixed for $L \rightarrow \infty$, $N_i \rightarrow \infty, 1 \leq i, \, j \leq n$.

The coupled mean-field integral equations for the densities of particles $\rho_i(\mathbf{x})$ ($1 \leq i \leq n$) take the form

$$\ln \rho_i(\mathbf{x}) = a_i + m_i \sum_{j=1}^{n} \frac{\eta_j}{m_j} \int \frac{\mathrm{d}^3 \mathbf{y}}{|\mathbf{y} - \mathbf{x}|} \, \rho_j(\mathbf{y}) \; , 1 \leq i \leq n \; , \tag{40}$$

where Lagrange multipliers $a_1, a_2, ..., a_n$ enforce the normalization of the densities.

$$\int \mathrm{d}^3 \mathbf{x} \, \rho_i(\mathbf{x}) = 1 \; , \; 1 \leq i \leq n \; . \tag{41}$$

Eqs.(40) give for the gravitational potential

$$U(\mathbf{x}) = -\frac{T}{m_1} \left[\Phi_1(\mathbf{x}) - a_1 \right] = -\frac{T}{m_2} \left[\Phi_2(\mathbf{x}) - a_2 \right] = ... = -\frac{T}{m_n} \left[\Phi_n(\mathbf{x}) - a_n \right] \; . \tag{42}$$

Setting,

$$\rho_i(\mathbf{x}) = \exp \left[\Phi_i(\mathbf{x}) \right] \; , \; 1 \leq i \leq n \tag{43}$$

and applying to eqs.(40) Laplace operator, we find the partial differential equations

$$\Delta \Phi_i(\mathbf{x}) + 4\pi \, m_i \sum_{j=1}^{n} \frac{\eta_j}{m_j} \, e^{\Phi_j(\mathbf{x})} = 0 \; , \; 1 \leq i, j \leq n. \tag{44}$$

Using eq.(42), we reduce eqs.(44) to a single equation

$$\Delta \Phi_1(\mathbf{x}) + 4\pi \, m_1 \sum_{j=1}^{n} \frac{\eta_j}{m_j} \, e^{a_j - \frac{m_j}{m_1} a_1} \, e^{\frac{m_j}{m_1} \Phi_1(\mathbf{x})} = 0 \; .$$

Eqs. (44) are scale covariant. If $\Phi_1, \Phi_2, ..., \Phi_n$ are solutions of eqs.(44), then $\Phi_{1\lambda}, \Phi_{2\lambda}, ..., \Phi_{n\lambda}$ defined by

$$\Phi_{i\lambda}(\mathbf{x}) = \Phi_i(\lambda \mathbf{x}) + \ln \lambda^2 \; , 1 \leq i \leq n \tag{45}$$

are also solutions of eq.(44). This property is due to the scale behaviour of Newton's potential.

In the spherically symmetric case the mean field equations (44) become ordinaries non-linear differential equations

$$\frac{\mathrm{d}^2 \Phi_i}{\mathrm{d}R^2} + \frac{2}{R} \frac{\mathrm{d}\Phi_i}{\mathrm{d}R} + 4\pi m_i \sum_{j=1}^{n} \frac{\eta_j}{m_j} \, e^{\Phi_j(R)} = 0 \; , \; 1 \leq i \leq n. \tag{46}$$

Using the scale covariance of the mean field eqs.(44) by the transformation (45), we can set

$$\Phi_i(R) = \chi_i\left(\lambda\frac{R}{R_{max}}\right) + \ln\left(\frac{\lambda^2}{3\eta_i^R}\right) \quad , \quad 1 \le i \le n \tag{47}$$

with new parameters $\eta_i^R = \frac{\eta_i}{R_{max}}$, $1 \le i \le n$. In this way, the mean field eqs.(46) become a reduced system of the form

$$\chi_i''(\lambda) + \frac{2}{\lambda}\chi_i'(\lambda) + m_i\sum_{j=1}^{n}\frac{e^{\chi_j(\lambda)}}{m_j} = 0 \quad , \quad 1 \le i \le n. \tag{48}$$

Let us find the boundary conditions for these equations. In order to have a regular solution at the origin we impose

$$\chi_i'(0) = 0 \quad , \quad 1 \le i \le n \ .$$

The system (48) is invariant under the transformation

$$\lambda \to \lambda\, e^\alpha, \ \chi_i \to \chi_i - 2\alpha \quad , \quad 1 \le i \le n\ .$$

Hence, we can choose

$$\chi_1(0) = 0$$

without loosing generality. As in the case of two kinds of particles, the remaining boundary conditions $\chi_2(0), ..., \chi_n(0)$ are not independent from $\eta_1, \eta_2, ..., \eta_n$. The normalization (41) of the densities of the n kinds of particles $\rho_1, \rho_2, \ldots, \rho_n$ has to be imposed. We obtain from eqs.(43) and (47),

$$\eta_i^R = \frac{1}{\lambda}\int_0^\lambda \mathrm{d}x\ x^2\ e^{\chi_i(x)} \quad , \quad 1 \le i \le n. \tag{49}$$

Using the reduced mean field equations (48), it is straightforward to show from eq.(49) that

$$m_i\sum_{j=1}^{n}\frac{\eta_j^R}{m_j} = -\lambda\,\chi_i'(\lambda) \quad , \quad 1 \le i \le n.$$

Hence,

$$\chi_j'(\lambda) = \frac{m_j}{m_i}\,\chi_i'(\lambda) \quad , \quad 1 \le i, j \le n. \tag{50}$$

From eq.(50) we introduce new λ-independent parameters

$$C_i = \chi_i(\lambda) - \frac{m_i}{m_1}\,\chi_1(\lambda) \quad , \quad 2 \le i \le n.$$

These new parameters are only function of $\eta_1^R, \eta_2^R, ..., \eta_n^R$. Notice that the boundary conditions can be written as,

$$\chi_2(0) = C_2, \quad \ldots \quad , \quad \chi_n(0) = C_n\ .$$

The reduced mean field equations (48) become a single ordinary differential equation with its coefficients depending on the parameters $C_2, ..., C_n$,

$$\chi_1''(\lambda) + \frac{2}{\lambda}\chi_1'(\lambda) + m_1\sum_{i=1}^{n}\frac{e^{C_i}}{m_i}\,e^{\frac{m_i}{m_1}\chi_1(\lambda)} = 0 \tag{51}$$

and the boundary conditions $\chi_1(0) = 0$, $\chi_1'(0) = 0$. [Here, $C_1 \equiv 0$].

Using eqs.(43) and (47) we can express the densities of the n kinds of particles in terms of the solution of eq.(51)

$$\rho_i(R) = \frac{\lambda^2 \, e^{C_i}}{3\eta_i^R} \, e^{\frac{m_i}{m_1}\chi_1(\lambda\frac{R}{R_{max}})} \quad , \quad 1 \le i \le n \ .$$

The thermodynamic functions are expressed as functions of the density of particles at the boundary $f_1, f_2, ..., f_n$ depending on $\eta_1^R, \eta_2^R, ..., \eta_n^R$. That is,

$$f_i = \frac{\lambda^2 \, e^{C_i}}{3\eta_i^R} \, e^{\frac{m_i}{m_1}\chi_1(\lambda)} \quad , \quad 1 \le i \le n \ .$$

We provide below the expressions for the free energy, the gravitational energy, the entropy and the equation of state.

$$\frac{F - F_0}{T} = \sum_{i=1}^{n} N_i \, [\, \ln f_i - \sum_{j=1}^{n} \frac{m_i}{m_j} \, \eta_j^R + 3(1 - f_i) \,]$$

$$E_P = 3T \sum_{i=1}^{n} N_i \, (f_i - 1) \tag{52}$$

$$S = S_0 + \sum_{i=1}^{n} N_i \, [\, 6(f_i - 1) - \ln f_i + \sum_{j=1}^{n} \frac{m_i}{m_j} \, \eta_j^R \,]$$

$$\frac{PV}{T} = \sum_{i=1}^{n} N_i \, f_i \tag{53}$$

where

$$F_0 = - \sum_{i=1}^{n} N_i \, T \, \ln \left[\frac{eV}{N_i} \left(\frac{m_i T}{2\pi} \right)^{\frac{3}{2}} \right]$$

is the free energy and

$$S_0 = \sum_{i=1}^{n} N_i \left(\ln \left[\frac{V}{N_i} \left(\frac{m_i T}{2\pi} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right)$$

is the entropy of the perfect gas with masses $m_1, \ m_2, ..., \ m_n$. Combining eqs.(52) and (53) yields the virial theorem

$$\frac{PV}{T} = \sum_{i=1}^{n} N_i + \frac{E_P}{3T} \ .$$

5   Conclusions

The self-gravitating gas with two kinds of particle has analogous qualitative properties to the self-gravitating gas with one kind of particles. Physical quantities like energy, free energy and entropy turn to be the sum of a term proportional to N_1 plus another term proportional to N_2 for large N_1, N_2 and V provided $N_1/V^{\frac{1}{3}}$ and $N_2/V^{\frac{1}{3}}$ are kept fixed. All physical quantities are expressed as functions of η_1 and η_2. Instead of a critical line as for one kind of particles, we have here a critical line in the (η_1, η_2) plane for the canonical ensemble . This line is associated to the Jeans instability.

The equation of state exhibits a rim on this critical line [see fig.3]. The thermodynamic functions exhibit a two-sheeted structure as functions of η_1^R and η_2^R. The branch points

are on the critical line. The specific heat is discontinuous and diverges there while the free energy is finite and continuous in the branch line.

The local pressure and the local densities of particles are related by the same equation as in a perfect gas [see sec. 3.5]:

$$P(\mathbf{r}) = \frac{N_1 T}{V} \rho_1(\mathbf{r}) + \frac{N_2 T}{V} \rho_2(\mathbf{r}) .$$

This can be explained by the dilute character of the self-gravitating gas in thermal equilibrium: $N/V \sim N^{-2} \to 0$ for $N \to \infty$. This dilution damps the effective interparticle interaction and allows a free particle behaviour.

The particle distribution is inhomogeneous and scales with R with a Haussdorf dimension d. The Haussdorf dimension d decreases for increasing η_1^R and η_2^R. Its value on the critical line $d = 1.6\ldots$ turns to be independent of the ratio η_1^R/η_2^R implying an universal behaviour. d takes there the same value than for the canonical critical point with one kind of particles [4].

.1   Appendix A

The goal of this appendix is to compute the expression

$$A = \frac{N_1}{2} \int d^3\mathbf{x} \, \Phi_1(\mathbf{x}) \, e^{\Phi_1(\mathbf{x})} + \frac{N_2}{2} \int d^3\mathbf{x} \, \Phi_2(\mathbf{x}) \, e^{\Phi_2(\mathbf{x})} .$$

Using eq.(9) we obtain

$$A = \frac{N_1}{2} \int d^3\mathbf{x} \, \Phi_1(\mathbf{x}) \left[e^{\Phi_1(\mathbf{x})} + \frac{m_2 N_2}{m_1 N_1} e^{\Phi_2(\mathbf{x})} \right] + \frac{N_2}{2} \left(a_2 - \frac{m_2}{m_1} a_1 \right) . \tag{54}$$

Using eq.(11) in the spherical symmetry the first term of eq.(54) becomes

$$\frac{N_1}{2} \int d^3\mathbf{x} \, \Phi_1(\mathbf{x}) \left(e^{\Phi_1(\mathbf{x})} + \frac{m_2 N_2}{m_1 N_1} e^{\Phi_2(\mathbf{x})} \right) =$$
$$-\frac{N_1}{2\eta_1} \int_0^{R_{max}} dR \, \Phi_1(R) \, \frac{d}{dR} (R^2 \frac{d\Phi_1}{dR}) . \tag{55}$$

Using eqs. (16) and (20) we obtain

$$\Phi_1'(R_{max}) = -\frac{\lambda}{R_{max}} (\eta_1^R + \mu \eta_2^R) . \tag{56}$$

Integrating by parts (55) and using eqs.(16) and (56) we obtain

$$\frac{N_1}{2} \int d^3\mathbf{x} \, \Phi_1(\mathbf{x}) \left[e^{\Phi_1(\mathbf{x})} + \frac{m_2 N_2}{m_1 N_1} e^{\Phi_2(\mathbf{x})} \right] = \tag{57}$$
$$\frac{N_1}{2} \left(1 + \frac{m_2 N_2}{m_1 N_1} \right) \ln f_1 + \frac{N_1}{2\lambda \eta_1^R} \int_0^\lambda dx \, x^2 \, [\chi_1'(x)]^2 .$$

Using eq.(29) the second term of (54) yields

$$\frac{N_2}{2} \left(a_2 - \frac{m_2}{m_1} a_1 \right) = \frac{N_2}{2} \left(\ln f_2 - \frac{m_2}{m_1} \ln f_1 \right) . \tag{58}$$

Using eqs. (57) and (58) we express A as

$$A = \frac{N_1}{2} \ln f_1 + \frac{N_2}{2} \ln f_2 + \frac{N_1}{2\lambda\eta_1^R} \int_0^\lambda dx \, x^2 \, [\chi_1^{'}(x)]^2 \, . \tag{59}$$

We compute now

$$I(\lambda) = \int_0^\lambda dx \, x^2 \, [\chi_1^{'}(x)]^2 \, .$$

Using eqs.(17) and (21), we derive the function

$$B(x) = x^3[e^{\chi_1(x)} + \mu^2 e^{\chi_2(x)}]$$

and find

$$B^{'}(x) = -x^3\chi_1^{'}(x)\chi_1^{''}(x) - 2x^2[\chi_1^{'}(x)]^2 + 3 \, x^2[e^{\chi_1(x)} + \mu^2 e^{\chi_2(x)}] \, . \tag{60}$$

We integrate $B^{'}$ between 0 and λ. Integrating by parts the first term and using eq.(20), we find

$$-\frac{\lambda}{2}(\eta_1^R + \mu \, \eta_2^R)^2 + \frac{3}{2}I(\lambda) \, .$$

The second term of eq.(60) yields

$$-2I(\lambda) \, .$$

Using eq.(19) the third term of (60) yields

$$3\lambda(\eta_1^R + \mu^2 \, \eta_2^R) \, .$$

Hence,

$$I(\lambda) = -2\lambda^3[e^{\chi_1(\lambda)} + \mu^2 e^{\chi_2(\lambda)}] - \lambda(\eta_1^R + \mu \, \eta_2^R)^2 + 6\lambda(\eta_1^R + \mu^2 \, \eta_2^R) \, .$$

Therefore, using eqs.(2) and (28) we obtain

$$\begin{aligned}
\frac{N_1}{2\lambda\eta_1^R} \int_0^\lambda dx \, x^2 \, [\chi_1^{'}(x)]^2 &= N_1 \left[3(1 - f_1) - \frac{1}{2}(\eta_1^R + \mu \, \eta_2^R)\right] \tag{61} \\
&= N_2 \left[3(1 - f_2) - \frac{1}{2}\left(\frac{1}{\mu} \, \eta_1^R + \eta_2^R\right)\right] \, .
\end{aligned}$$

Hence, the expression of A [using eqs.(59) and (61)] is

$$\begin{aligned}
A &= N_1 \left[\frac{1}{2}\ln f_1 + 3(1 - f_1) - \frac{1}{2}\left(\eta_1^R + \mu \, \eta_2^R\right)\right] \\
&+ N_2 \left[\frac{1}{2}\ln f_2 + 3(1 - f_2) - \frac{1}{2}\left(\frac{1}{\mu} \, \eta_1^R + \eta_2^R\right)\right] \, .
\end{aligned}$$

.2   Appendix B

We derive here the mean field equations from the hydrostatic equilibrium condition combined with the Poisson equation for a mixture of two ideal gases.

The hydrostatic equilibrium condition [6]

$$\nabla P(\vec{q}) = -\left[m_1 \, N_1 \, \rho_1(\vec{q}) + m_2 \, N_2 \, \rho_2(\vec{q})\right] \, \nabla U(\vec{q}) \, ,$$

where $P(\vec{q})$ stands for the pressure, combined with the equation of state for the ideal gas in local form

$$P(\vec{q}) = T \left[N_1 \, \rho_1(\vec{q}) + N_2 \, \rho_2(\vec{q})\right] \, ,$$

yields for the particle densities

$$\rho_1(\vec{q}) = \rho_1^0 \, e^{-\frac{m_1}{T} U(\vec{q})} \quad , \quad \rho_2(\vec{q}) = \rho_2^0 \, e^{-\frac{m_2}{T} U(\vec{q})}$$

where ρ_1^0 and ρ_2^0 are constants. Inserting this relation into the Poisson equation

$$\nabla^2 U(\vec{q}) = 4\pi G \left[m_1 \, N_1 \, \rho_1(\vec{q}) + m_2 \, N_2 \, \rho_2(\vec{q}) \right]$$

yields in dimensionless coordinates eq.(14).

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